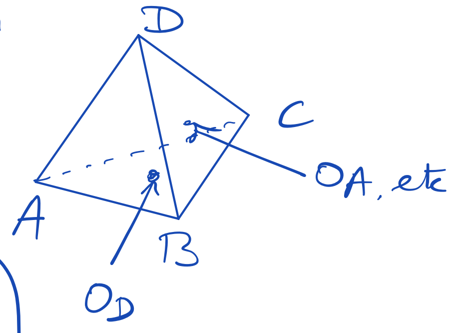
Answer model resit Symmetry in Physics June 29, 2023

1. Methane molecule CH_4 — rotational symmetry gp "G_M"

1a. G_M also symmetry gp of tetrahedron



120° rotations around DO_D : c, c^2
+ 3 analogous ones around AO_A, BO_B, CO_C } $4 \times 120°$ rotations, $4 \times 240°$ "

$c^3 = e$ ← 0° rotation

180° rotations around axis through midpoints of DC & AB : b
+ 2 analogous ones } $3 \times 180°$ rotation

So $\underset{\uparrow e}{1} + 8 + 3 = 12$ elements

$G_M = A_4$
(gp of even permutations of 4 numbers)

1b. Conjugacy classes : $(e) = \{e\}$

$c \not\sim c^2$ no reflection in group

so (c) 8 (c^2) & (b) ← all related by rotation.

4 classes
‖
4 irreps

use $\sum_\mu n_\mu^2 = [g]$ sum over irreps

$1 + n_2^2 + n_3^2 + n_4^2 = 12$

$\Rightarrow \begin{cases} n_2 = n_3 = 1 \\ n_4 = 3 \end{cases}$

	(e)	(c)	(c^2)	(b)
$D^{(1)}$	1	1	1	1
$D^{(2)}$	1	ω	ω^2	1
$D^{(3)}$	1	ω^2	ω	1
$D^{(4)}$	3	0	0	-1

character table not requested in exercise

Answer for A_4 also correct: $(e)=\{e\}$, $((123))=\{(123), \dots\}$
$((132))=\{(132), \dots\}$, $((12)(34))=\{(12)(34), \dots\}$

1c characters of D^V of subgp of $SO(3)$:

$$\chi^V(\theta) = 1+2\cos\theta$$

$$\Rightarrow \chi^V(e)=3, \quad \chi^V(c)=0, \quad \chi^V(c^2)=0,$$

$$\chi^V(b)=-1$$

$$\langle \chi^{(1)}, \chi^V \rangle = \frac{1}{12}\left(1\cdot 3 + 0 + 0 + 3\cdot -1\right) = 0$$

$\Rightarrow$ no invariant vector allowed in system with S_4 symmetry.

2. $O(3)$ action on σ_{ij}, D^V defining or vector irrep of $O(3)$

2a. Show that $\sigma \xrightarrow{g} \sigma' = D^V(g)\,\sigma\, D^V(g)^T$ for $g \in O(3)$ by writing out indices on example $\sigma_{ij} = x_i y_j$

$$x_i \to x_k' = D^V_{ki} x_i \qquad \text{idem } y_j \to y_\ell' = D^V_{\ell j} y_j$$

$$\sigma_{ij} = x_i y_j \longrightarrow \sigma_{k\ell}' = x_k' y_\ell' = D^V_{ki} D^V_{\ell j} x_i y_j$$

$$= D^V_{ki} \sigma_{ij} D^V_{\ell j} = D^V_{ki} \sigma_{ij} D^{VT}_{j\ell}$$

$$\Rightarrow \sigma' = D^V \sigma D^{VT}$$

2b. If $\sigma = \sigma' \Rightarrow \sigma = D^V \sigma D^{VT}$

75%

$$\Rightarrow \sigma D^V = D^V \sigma \underbrace{D^{VT} D^V}_{\mathbb{1}} = D^V \sigma$$

$$\Rightarrow [\sigma, D^V] = 0$$

25% D^V irrep therefore Schur's lemma implies $\sigma = \lambda \mathbb{1}$

is only invariant tensor under $O(3)$ trafo's

2c. Determine subgp of $O(3)$ trafo's that leave

$$\sigma_{ij} = \delta_{ij} + a(\delta_{i2}\delta_{j2} + \delta_{i3}\delta_{j3}) \text{ invariant}$$

for nonzero constant a. i.e. $\sigma = \begin{pmatrix} 1 & & \\ & 1+a & \\ & & 1+a \end{pmatrix}$

$$[\sigma, D^V] = 0 \quad \Rightarrow \quad D^V = \begin{pmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{pmatrix}$$

D^V is in $O(3)$ if $\begin{pmatrix} e & f \\ h & i \end{pmatrix} \in O(2)$ and $a = \pm 1$

hence

$$D^V = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & & \\ 0 & O_{2\times 2} & \end{pmatrix} \qquad O_{2\times 2} \in O(2)$$

$\Rightarrow$ All rotations around x-axis (in y-z plane)
all reflections in the x-y, x-z, y-z planes
and all their products. (Description suffices)

3. Galilei trafo's in 1+1 D: $x \to x' = x + vt + a$
$t \to t' = t + b$.

3a. $T(a,b,v)$ acting on $\begin{pmatrix} x \\ t \\ 1 \end{pmatrix}$: $\begin{pmatrix} 1 & v & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$

reducible since $\left(\begin{array}{c|c} & \\ \hline 0 & \end{array}\right)$ in other words $\begin{pmatrix} x \\ t \\ 0 \end{pmatrix}$ invariant subspace

3b $T(a,b,v)\,T(c,d,v') = \begin{pmatrix} 1 & v & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & v' & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} =$

$$= \begin{pmatrix} 1 & v+v' & a+c+vd \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{pmatrix} = T(a+c+vd,\ b+d,\ v+v')$$

T^{-1} is such that $T(a,b,v)\,T^{-1}(a,b,v) = \underline{1\!\!1}$.

$$\underline{1\!\!1} = T(0,0,0)$$

$$\Rightarrow \left.\begin{aligned} a+c+vd &= 0 \\ b+d &= 0 \\ v+v' &= 0 \end{aligned}\right\} \Rightarrow d = -b \ \& \ v' = -v$$

$$\Rightarrow c = -a + vb$$

$$T^{-1} = T(-a+vb,\ -b,\ -v)$$

3c. no 2D matrix rep with translations represented nontrivially because trafo is nonlinear & 2D matrix rep is linear trafo.

D. Boer

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